Exercise 1. The weak derivative of f is given by

$$f'(x) = \begin{cases} 1 & \text{for all } x > 0 \\ -1 & \text{for all } x < 0. \end{cases}$$

Indeed, for all $\varphi \in \mathscr{D}(I) = C_c^\infty(I)$, we have

$$\begin{aligned} \int_{-1}^1 f(x)\varphi'(x)dx &= -\int_{-1}^0 x\,\varphi'(x)dx + \int_0^1 x\,\varphi'(x)dx \\ &= -\left[x\,\varphi(x)\right]_0^R + \int_{-1}^0 \varphi(x)dx + \left[x\,\varphi(x)\right]_0^R - \int_0^1 \varphi(x)dx \\ &= -\int_{-1}^1 f'(x)\varphi(x)dx. \end{aligned}$$

By contradiction, assume that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $W^{1,\infty}$. As $\{f_n\}_{n\in\mathbb{N}}$ and $\{f_n'\}_{n\in\mathbb{N}}$ converge uniformly, we deduce in particular that $f \in C^1(I)$, which contradicts the fact that f is not differentiable at $x = 0$.

Exercise 2.

1. By the co-area formula, if $\beta(d) = \mathscr{H}^{d-1}(S^{d-1}) = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the area of the $(d-1)$-dimensional sphere $S^{d-1} \subset \mathbb{R}^d$, we have

$$\int_{B(0,R)} |u(x)|^p dx = \beta(d) \int_0^R r^{d-1}|f(r)|^p dr,$$

which proves the claim.

2. We have

$$\nabla u(x) = \frac{x}{|x|} f'(|x|), \tag{1}$$

which shows that $|\nabla u(x)| = |f'(|x|)|$, and the claim follows from the previous formula, provided that we show that the expression (1) is the weak derivative of u as a Sobolev function. For all $\varphi \in C_c^\infty(B(0,R))$, we have

$$\int_{B_R\setminus\overline{B}_\varepsilon(0)} u\,\frac{\partial\varphi}{\partial x_i}dx = -\int_{B_R\setminus\overline{B}_\varepsilon(0)} \varphi\,\frac{\partial u}{\partial x_i}dx - \int_{\partial B(0,\varepsilon)} u\,\varphi\frac{x_i}{|x|}d\mathscr{H}^{d-1}.$$

Assuming that

$$\int_0^R r^{d-1}|f'(r)|^p dr < \infty,$$

we deduce that $\dfrac{\partial u}{\partial x_i} \in L^p(B(0,R))$. Furthermore, as $u\,\dfrac{\partial\varphi}{\partial x_i} \in L^1(B(0,R))$, we deduce that

$$\lim_{\varepsilon\to 0}\int_{B_R\setminus\overline{B}_\varepsilon(0)} u\,\frac{\partial\varphi}{\partial x_i}dx = \int_{B_R(0)} u\,\frac{\partial\varphi}{\partial x_i}dx.$$

Furthermore, we have

$$\left|\int_{\partial B(0,\varepsilon)} u\,\varphi\,\frac{x_i}{|x|}d\mathscr{H}^{d-1}\right| \leq \beta(d)\,\|\varphi\|_{\mathrm{L}^\infty(B(0,R))}\,\varepsilon^{d-1}f(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Now, assuming that $u \in W^{1,p}(B(0,R))$, then the weak derivative of the restriction of u to $B(0,R)$ is given by the formula (1). The previous computation shows that ∇u satisfies the same integration by parts formula as the weak derivative and the fundamental lemma of the calculus of variations implies that functions that agree almost everywhere in $B(0,R) \setminus \{0\}$ agree almost everywhere in $B(0,R)$. Therefore, the expression (1) is the weak derivative and the proof is complete.

3. We have

$$\int_0^R r^{d-1}|f(r)|^p dr = \int_0^R r^{d-1+p\,\gamma} < \infty \Longleftrightarrow \gamma > -\frac{d}{p}. \tag{2}$$

Therefore, $u \in L^p(B(0,R))$ if and only if $\gamma > -\dfrac{d}{p}$. Likewise, since $f'(r) = \gamma\, r^{\gamma-1}$, we have $u \in W^{1,p}(B(0,R))$ if and only if $\gamma > 1 - \dfrac{d}{p}$.

Exercise 3. The proof is given in the lecture notes (Lemma 2.6.2).

Exercise 4. The proof follows the same approach as in the course for the Poincaré inequality. By contradiction, there exists $\{u_n\}_{n\in\mathbb{N}^*} \subset S$ is a sequence such that $\|u_n\|_{\mathrm{L}^p(\Omega)} \geq n\,\|\nabla u_n\|_{\mathrm{L}^p(\Omega)}$, up to replacing u_n by $\dfrac{u_n}{\|u_n\|_{\mathrm{L}^p(\Omega)}}$, we can assume that

$$\|u_n\|_{\mathrm{L}^p(\Omega)} = 1.$$

In particular, we have

$$\|\nabla u_n\|_{\mathrm{L}^p(\Omega)} \leq \frac{1}{n}\,\|u_n\|_{\mathrm{L}^p(\Omega)} = \frac{1}{n} \leq 1.$$

Therefore, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$, and since Ω is open, bounded, and Lipschitz, the theorem of Rellich-Kondrachov shows that there exists $u_\infty \in W^{1,p}(\Omega)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u_\infty$ strongly in $L^p(\Omega)$ and $\nabla u_n \underset{n\to\infty}{\rightharpoonup} \nabla u_\infty$ weakly in $L^p(\Omega)$. In particular, we have

$$\|\nabla u_\infty\|_{\mathrm{L}^p(\Omega)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{\mathrm{L}^p(\Omega)} = 0.$$

Therefore we have $\nabla u_\infty = 0$, and since $\|\nabla u_n\|_{\mathrm{L}^p(\Omega)} \underset{n\to\infty}{\longrightarrow} 0$, we deduce that

$$\lim_{n\to\infty} \|u_n - u_\infty\|_{\mathrm{W}^{1,p}(\Omega)} = 0.$$

Since S is closed, we have $u_\infty \in S$. However, the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$ shows that

$$\|u_\infty\|_{\mathrm{L}^p(\Omega)} = \lim_{n\to\infty} \|u_n\|_{\mathrm{L}^p(\Omega)} = 1. \tag{3}$$

Since $\nabla u_\infty = 0$ and $u_\infty \in S$, we must have $u_\infty = 0$, and that contradicts (3).

For example, if $\alpha > 0$ and $S = W^{1,p}(\Omega) \cap \big\{u : \mathscr{L}^d(\Omega \cap \{x : |u(x)| \neq 0\}) \geq \alpha\big\}$, then we get a Poincaré inequality for this class of functions.